12. Schemes

Have seen:

$$\left\{\begin{array}{l}\text{affine varieties } X \\ \text{over } K=\overline{K}\end{array}\right\} \overset{1:1}{\underset{X \mapsto A(X)}{\longleftrightarrow}} \left\{\begin{array}{c}\text{finitely generated reduced} \\ K\text{-algebras}\end{array}\right\}$$

Algebraic geometry $\leftsquigarrow$ Commutative Algebra

Main problem

Solutions of equations over more general rings?

$$(\underbrace{\mathbb{Z}}_{\text{no } K\text{-algebra}},\ \underbrace{\mathbb{F}_p,\ \mathbb{R}}_{\text{not alg. closed}},\ \underbrace{K[x]/\langle x^2\rangle}_{\text{not reduced}},\ \underbrace{K[\![x]\!]}_{\text{not finitely generated}}, \ldots)$$

Plan of action

- Define geometric object associated to any ring R
  → affine scheme
- Give it more structure (Set $\rightsquigarrow$ top. space $\rightsquigarrow$ ringed space)
- Gluing construction $\rightsquigarrow$ more general geometric objects
  → schemes

Note:

- last chapters of course introduce you to language of modern algebraic geometry
- focus more on theory, less on applications

Sneak peak

$R = K[x]/\langle x^2\rangle \rightsquigarrow X = \mathrm{Spec}(R)$ $\quad \overset{R}{\bullet}$ ← ringed space $X = \{pt\}$, $\mathcal{O}_X(X) = R$

Fact Y variety $\rightsquigarrow \mathrm{Mor}(X,Y) \cong \{(a,v) : a \in Y,\ v \in T_a Y\}$

Def (Affine schemes)
Let R be a ring. Then the set

$$\operatorname{Spec} R = \{ p \trianglelefteq R : p \text{ prime ideal} \}$$

is called the <u>spectrum of R</u> or <u>affine scheme</u> assoc. to R.

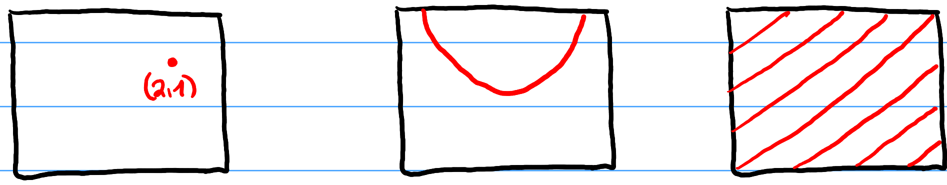
Exa
(a) $R = A[X]$, X affine variety over $K = \overline{K}$
Have:

$$X \xrightarrow[\sim]{1:1} \{\text{max. ideals } m \leq R\} \subseteq \operatorname{Spec} R$$
$$a \longmapsto m_a = I_X(a)$$

[Rmk. 2.9]

$$\operatorname{Spec} R \cong \{ Y \subseteq X : Y \text{ irred. subvariety of } X \}$$

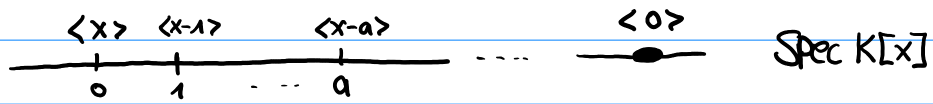
E.g.



$\operatorname{Spec}(K[x_1,x_2]) \ni \langle x_1-2, x_2-1 \rangle$, $\langle x_2 - x_1^2 \rangle$, $\langle 0 \rangle$

(b) $R = K[x] = A(\mathbb{A}^1_K)$, $K = \overline{K}$

$$\rightsquigarrow \operatorname{Spec} R = \{ \langle x-a \rangle : a \in K \} \cup \{ \langle 0 \rangle \}$$



$R = \mathbb{R}[x] \rightsquigarrow \operatorname{Spec} R \ni \langle x^2+1 \rangle \leftarrow$ pair of cplx conjug. points $\{\pm i\}$ in $\mathbb{A}^1_{\mathbb{C}}$

(c) $R = \mathbb{Z}$
$$\rightsquigarrow \operatorname{Spec} R = \{ \langle p \rangle : p \text{ prime} \} \cup \{ \langle 0 \rangle \}.$$

More pictures: [Eisenbud-Harris: The Geometry of Schemes, Chapter II]

Rmk R ring, $\mathfrak{J} = \sqrt{\langle 0 \rangle}$ nilradical

$f \in \mathfrak{J}$, $\mathfrak{p} \in \operatorname{Spec}(R) \rightsquigarrow \exists m \in \mathbb{N} : f^m = 0 \in \mathfrak{p}$

$\overset{\mathfrak{p} \text{ prime}}{\rightsquigarrow} f \in \mathfrak{p}$

$\Longrightarrow$ For $\varphi : R \longrightarrow R/\mathfrak{J}$ quotient map:

$$\begin{aligned} \operatorname{Spec}(R/\mathfrak{J}) &\xrightarrow{\sim} \operatorname{Spec}(R) \quad \text{bijective.} \\ \mathfrak{q} &\longmapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

Lem prime ideals $\mathfrak{q} \trianglelefteq R/\mathfrak{J}$
$\widehat{=}$ prime ideals $\mathfrak{p} \trianglelefteq R$
containing $\mathfrak{J}$

Exa $\operatorname{Spec}(\underbrace{K[x]/\langle x^2 \rangle}_{\mathfrak{J} = \langle x \rangle}) \xrightarrow{1:1} \operatorname{Spec}(K[x]/\langle x \rangle) = \operatorname{Spec} K = \{\langle 0 \rangle\}.$

$\rightsquigarrow$ set (or top. space) $\operatorname{Spec}(R)$ cannot detect nilpot. elements

$\rightsquigarrow$ ringed spaces $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R/\mathfrak{J})$ will be different !

Functions and vanishing loci on affine schemes

Before · X affine variety $\rightsquigarrow$ $A(X)$ ring of coord. functions on X

· $f \in A(X)$, $x \in X$ $\rightsquigarrow$ evaluation $f(x) \in K$

· $I \trianglelefteq A(X) \rightsquigarrow$ vanishing locus $V_X(I) \subseteq X$

how to generalize? to $X = \operatorname{Spec} R$

Idea Functions on $\operatorname{Spec}(R)$ $\hat{=}$ elements $f \in R$

$\rightsquigarrow$ how to "evaluate at $p \in R$"?

Def (Functions on affine schemes)

R ring, $p \in \operatorname{Spec} R$

(a) We define the residue field of $\operatorname{Spec} R$ at p as

$$K(p) = \operatorname{Frac}(R/p)$$

fraction field (localiz. at $\langle 0 \rangle \subseteq R/p$) integral domain

(b) For $f \in R$ we define the value of f at p as

$$f(p) := \frac{[f]}{1} \in K(p)$$ $\leftarrow$ image of f under $R \to R/p \to K(p)$

$\rightsquigarrow f(p) = 0 \iff f \in p$

Exa (a) $R = A(X)$, X affine variety $/ K = \overline{K}$

· $p = m_a = I_X(a)$ for $a \in X$

$\rightsquigarrow R/p \xrightarrow{\sim} K$, $[f] \mapsto f(a)$ $\rightsquigarrow$ $K(p) = \operatorname{Frac}(K) = K$ and $f(a) =$ value of f at a.

· $p = I_X(Y)$ for $Y \subseteq X$ irred. subvariety

$\rightsquigarrow R/p \cong A(Y)$ and $K(p) = \operatorname{Frac}(A(Y)) = K(Y)$

[Rmk. 9.8(b)] rat'l funct. $Y \dashrightarrow \mathbb{A}^1_K$

(b) $R = \mathbb{Z}$, $f \in \mathbb{Z}$

· $p = \langle p \rangle$, p prime $\rightsquigarrow K(p) = R/p = \mathbb{Z}/p\mathbb{Z} \rightsquigarrow f(p) = [f] \in \mathbb{Z}/p\mathbb{Z}$

· $p = \langle 0 \rangle$ $\rightsquigarrow K(p) = \operatorname{Frac}(\underbrace{R/p}_{=\mathbb{Z}}) = \mathbb{Q} \rightsquigarrow f(p) = \frac{f}{1} \in \mathbb{Q}$

Def (Zero loci and ideals in affine schemes)
R ring
(a) $S \subseteq R \rightsquigarrow$ zero locus (or vanishing locus) of S

$$V(S) = \{ p \in \operatorname{Spec} R : f(p) = 0 \ \forall f \in S \} \subseteq \operatorname{Spec} R$$

(b) $X \subseteq \operatorname{Spec} R \rightsquigarrow$ ideal of X

$$I(X) = \{ f \in R : f(p) = 0 \ \forall p \in X \} \trianglelefteq R$$

ideal: evaluation $R \to K(p)$ is ring homomorph.

Rmk
(a) $f(p) = 0 \iff f \in p$

$$\rightsquigarrow V(S) = \{ p \in \operatorname{Spec} R : f \in p \ \forall f \in S \} = \{ p \in \operatorname{Spec} R : p \supseteq S \}$$

$$I(X) = \bigcap_{p \in X} p$$

(b) Why have $f(p) \in K(p)$ and not in R/p?
$\rightsquigarrow$ later: regular fcts $\varphi =$ locally g/f
$\rightsquigarrow \varphi(p) = g(p)/f(p) \in K(p)$.

values of regular fcts. live in same ring as values of global functions $f \in R$.

(c) Basic properties of $V(-)$ and $I(-)$ same as before:

- $V(S) = V(\langle S \rangle)$
- $V(S_1) \cup V(S_2) = V(S_1 S_2)$
- $\bigcap_{i \in J} V(S_i) = V(\bigcup_{i \in J} S_i)$
- $V(1) = \emptyset$
- $V(0) = \operatorname{Spec} R$.

The Zariski topology & scheme-theoretic Nullstellensatz

Def (Zariski topology)
Endow the affine scheme $\operatorname{Spec} R$ with topology

$$\mathcal{C} = \{ \underset{\uparrow}{V(S)} \subseteq \operatorname{Spec} R : S \subseteq R \}$$

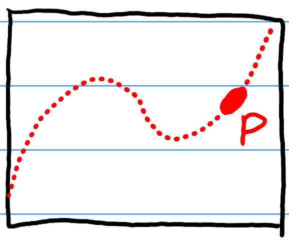
closed sets

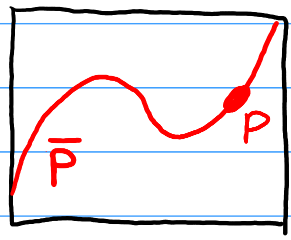
Rmks
(a) Automatically get: connectedness, irreducibility, dimension for schemes
(b) Zariski topology is not T_1: points not necess. closed!
In fact: $p \in \operatorname{Spec}(R)$

$$\rightsquigarrow \overline{\{p\}} = \bigcap_{\underbrace{V(S) \ni p}_{\Leftrightarrow\, p \supseteq S}} V(S) = V\Big(\bigcup_{p \supseteq S} S\Big) = V(p) = \{ q \in \operatorname{Spec} R : q \supseteq p \}$$

$\rightsquigarrow$ $\{p\}$ is closed $\Leftrightarrow$ p maximal ideal

Exa $R = K[x_1, x_2]$, $p = \langle x_2 - x_1^3 + x_1 \rangle \longleftrightarrow Y = V(p) \subseteq \mathbb{A}^2_K$





$\overline{p} = \{p\} \cup \{ \langle x_1 - a_1, x_2 - a_2 \rangle : (a_1, a_2) \in Y \}$

In general X aff. var., $R = A(X)$, $Y \subseteq X$ irred. subvariety
$\Rightarrow$ $p = I_X(Y) \in \operatorname{Spec} R$: generic point of Y

$f \in K(p) = K(Y)$: rational fct. on Y
$\rightsquigarrow$ defined at general point of Y

Pro (Scheme-theoretic Nullstellensatz)
R ring
(a) For $X \subseteq \operatorname{Spec} R$ closed, we have $V(I(X)) = X$.
(b) For $J \trianglelefteq R$ ideal, we have $I(V(J)) = \sqrt{J}$.
In particular:

$$\left\{\begin{array}{c}\text{closed subsets}\\ \text{of } \operatorname{Spec} R\end{array}\right\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \left\{\begin{array}{c}\text{radical ideals}\\ \text{in } R\end{array}\right\}, \quad \sim$$

Pf Same procedure as every year, except (b) "$\subseteq$":

$$I(V(J)) = I\big(\{p : p \supseteq J\}\big) = \bigcap_{\substack{p \in \operatorname{Spec} R\\ p \supseteq J}} p \underset{\text{Commut. Algebra}}{=} \sqrt{J}$$
$\square$

Rmk (Properties of $V(-)$ and $I(-)$)
(a) $J_1, J_2 \trianglelefteq R$

$\rightsquigarrow$ $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$
$V(J_1) \cap V(J_2) = V(J_1 + J_2)$

(b) $X_1, X_2 \subseteq \operatorname{Spec} R$ closed

$\rightsquigarrow$ $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$
$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

Pf Same as before, using Nullstellensatz. $\square$

Distinguished opens in affine schemes

Def (Distinguished open subsets)
R ring, $f \in R$

$$D(f) = \mathrm{Spec}(R) \setminus V(f) = \{ p \in \mathrm{Spec}\, R : f \notin p \}$$

is called the <u>distinguished open subset</u> of f in $\mathrm{Spec}\, R$.

Rmks
(a) X aff. variety $/ K = \overline{K}$ $\rightsquigarrow$ $\Phi : X \hookrightarrow \mathrm{Spec}\, A(X)$, $a \longmapsto m_a = I_X(a)$

$\Rightarrow$ Φ injective, $\Phi(X) =$ set of closed pts in $\mathrm{Spec}\, A(X)$

$\Phi^{-1}(V(J)) = \{ a \in X : m_a \supseteq J \} = V_X(J) \subseteq X \quad \forall J \trianglelefteq A(X)$

$\Phi^{-1}(D(f)) = \{ a \in X : m_a \not\ni f \} = D(f) \subseteq X \quad \forall f \in A(X)$

$\Rightarrow$ Φ is homeomorphism onto its image.

(b) $U \subseteq \mathrm{Spec}\, R$ open $\rightsquigarrow$ $U = \mathrm{Spec}\, R \setminus V(S)$

$$\Rightarrow U = \mathrm{Spec}\, R \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} \underbrace{\mathrm{Spec}\, R \setminus V(f)}_{D(f)}$$

$\Rightarrow$ $\{ D(f) : f \in R \}$ basis of Zariski topology on $\mathrm{Spec}\, R$

Lem (Partition of unity reloaded)
$D(f) \overset{(*)}{=} \bigcup_{i \in I} D(f_i) \Rightarrow \exists I_0 \subseteq I$ finite, $r_i \in R$ $(i \in I_0)$, $m \in \mathbb{N}$: $f^m = \sum_{i \in I_0} r_i f_i \in R$

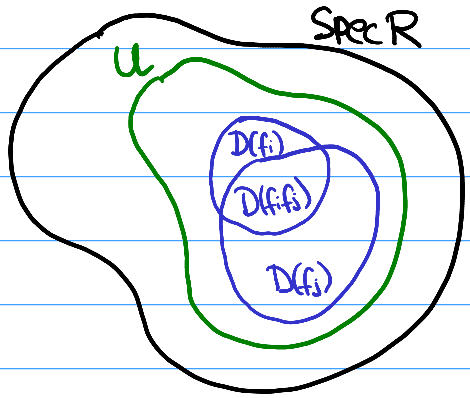
Pf $(*) \Leftrightarrow V(f) = \bigcap_{i \in I} V(f_i) = V(\langle f_i : i \in I \rangle)$

Nullstellensatz : $f \in \sqrt{\langle f \rangle} = \sqrt{\langle f_i : i \in I \rangle}$

$\Rightarrow f^m \in \langle f_i : i \in I \rangle$ $\square$

Regular functions & the structure sheaf of affine schemes

Question For $U \subseteq \operatorname{Spec} R$ open, what are reg. functions on U?



Want · $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$

· $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$

· $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf

$\Rightarrow$ This is enough to know $\mathcal{O}_{\operatorname{Spec} R}(U)$!

Indeed Choose cover $\{D(f_i) : i \in I\}$ of U

$\rightsquigarrow$ $\varphi \in \mathcal{O}_{\operatorname{Spec} R}(U)$ uniquely determined by $\varphi_i \in \mathcal{O}_{\operatorname{Spec} R}(D(f_i)) = R_{f_i}$ s.t.

$$\varphi_i|_{D(f_if_j)} = \varphi_j|_{D(f_if_j)} \text{ in } R_{f_if_j}$$

Good: this uniquely determines $\mathcal{O}_{\operatorname{Spec} R}(U)$

Bad: · Why independent of choices?

· not local!

Def (Regular functions)

R ring, $U \subseteq \operatorname{Spec} R$ open. A <u>regular function</u> on U is a family

$$\varphi = (\varphi_p)_{p \in U} \quad \text{with } \varphi_p \in R_p \text{ for all } p \in U$$

$$\left.\begin{array}{l} \text{such that: } \forall p \in U \ \exists f, g \in R \text{ and } p \in \underbrace{U_p}_{\text{open}} \subseteq U: \\ f \notin q \text{ and } \varphi_q = \frac{g}{f} \in R_q \quad \forall q \in U_p \end{array}\right\} (*)$$

$\rightsquigarrow$ $\mathcal{O}_{\operatorname{Spec} R}(U)$ = set of φ above

$\rightsquigarrow$ ring structure: $\left.\begin{array}{l} \varphi + \psi = (\varphi_p + \psi_p)_{p \in U} \\ \varphi \cdot \psi = (\varphi_p \cdot \psi_p)_{p \in U} \end{array}\right\}$ condition $(*)$ works!

$\rightsquigarrow$ condition $(*)$ is local $\Rightarrow$ $\underbrace{\mathcal{O}_{\operatorname{Spec} R}(-)}_{\text{structure sheaf}}$ sheaf of rings

Rmk $R_p / p \cdot R_p \cong (R/p)_{pR/p} \cong \mathrm{Frac}(R/p) = K(p)$

$\rightsquigarrow$ For $\varphi \in \mathcal{O}_{\mathrm{Spec} R}(U)$ w/ $p \in U \rightsquigarrow \underbrace{\varphi(p)}_{\varphi_p \in R_p} \in K(p)$ value at p

Warning $(\varphi(p))_{p \in U}$ does not determine φ in general!

Lem (Stalks of regular functions)
R ring, $p \in \mathrm{Spec}\, R$. Then the map

$$\mathcal{O}_{\mathrm{Spec} R, p} \xrightarrow{\Phi} R_p, \ [(U, \varphi)] \longmapsto \varphi_p$$

is an isomorphism.

Pf Inverse map is given by

$$R_p \xrightarrow{\Psi} \mathcal{O}_{\mathrm{Spec} R, p}, \quad \frac{g}{f} \longmapsto [(D(f), \varphi = (\tfrac{g}{f})_{q \in D(f)}]$$

$f \notin p \rightsquigarrow p \in D(f)$ $\quad$ $q \in D(f) \Rightarrow f \notin q \Rightarrow \frac{g}{f} \in R_q$

$\Phi \circ \Psi = \mathrm{id}$: clear ✓
$\Psi \circ \Phi = \mathrm{id}$:

$$[(U, \varphi)] \in \mathcal{O}_{\mathrm{Spec} R, p} \overset{\text{def}}{\Longrightarrow} \exists\, p \in U_p \subseteq U : \varphi_q = \frac{g}{f} \ \forall q \in U_p$$

$$\Rightarrow [(U, \varphi)] = [(U_p, (\tfrac{g}{f})_{q \in U_p})] = \Psi(\tfrac{g}{f}) = \Psi(\Phi([(U_p, (\tfrac{g}{f})_q)]))$$

$\square$

Note $\mathcal{F}$ sheaf on X, $U \subseteq X$ open

$\Rightarrow \varphi \in \mathcal{F}(U)$ uniquely determ. by $(\varphi_p \in \mathcal{F}_p)_{p \in U}$

$\rightsquigarrow$ this is idea behind defining $\varphi \in \mathcal{O}_{\mathrm{Spec} R}(U)$ as $(\varphi_p)_{p \in U}$ + extra condit.

Regular functions on distinguished open sets

Pro R ring, $f \in R$. Then the map

$$R_f \xrightarrow{\Phi} \mathcal{O}_{\operatorname{Spec} R}(D(f)), \quad \frac{g}{f^m} \mapsto \varphi = \left(\frac{g}{f^m} \in R_p\right)_{p \in D(f)}$$

is an isomorphism. $\xrightarrow{f=1} \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R.$

Pf $\boxed{\Phi \text{ injective}}$ $\frac{g}{f^m} \in \ker(\Phi) \Rightarrow \frac{g}{f^m} = 0 \in R_p \ \forall p \in D(f)$

$\Rightarrow \exists h_p \in R \setminus p : g \cdot h_p = 0 \in R \quad (\star)$

Need to show: $\exists r \in \mathbb{N}: \ f^r g = 0 \Leftrightarrow f^r \in J = \underbrace{\operatorname{Ann}_R(g)}_{=\{h \in R : \, hg = 0 \in R\}}$

Claim $V(J) \subseteq V(f)$. Nullstellensatz

Pf of claim

$$V(J) = \{p \in \operatorname{Spec} R : p \supseteq J\}$$

For $p \in D(f)$: $(\star) \Rightarrow h_p \in J \setminus p \Rightarrow J \not\subseteq p \Rightarrow p \notin V(J)$

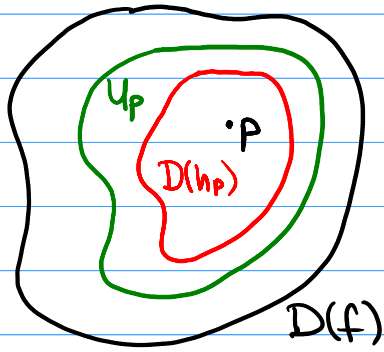
$\rightsquigarrow D(f) \subseteq \operatorname{Spec} R \setminus V(J) \xrightarrow[\text{in } \operatorname{Spec} R]{\text{Complement}} V(f) \supseteq V(J).$ ※

$\boxed{\Phi \text{ surjective}}$ $\varphi \in \mathcal{O}_{\operatorname{Spec} R}(D(f))$

$\forall p \in D(f) \ \exists h_p \in R : p \in D(h_p)$

$$\varphi = \frac{g_p}{f_p} \text{ on } D(h_p)$$

$\rightsquigarrow f_p \notin q \ \forall q \in D(h_p) \Leftrightarrow D(h_p) \subseteq D(f_p)$



Claim can assume $f_p = h_p$

Pf $D(h_p) \subseteq D(f_p) \Leftrightarrow V(h_p) \supseteq V(f_p) \xRightarrow{I(-)} h_p \in \sqrt{\langle h_p \rangle} \subseteq \sqrt{\langle f_p \rangle}$

$\Rightarrow h_p^r = c \cdot f_p$ for some $r \in \mathbb{N}$, $c \in R$

$\Rightarrow \dfrac{g_p}{f_p} = \dfrac{c \cdot g_p}{h_p^r}$ $\}$ new g_p $\}$ new $f_p = h_p^r \rightsquigarrow$ does not change $D(h_p)$

Summary $D(f)$ covered by $D(f_p)$ and $\varphi = \frac{g_p}{f_p}$ on $D(f_p)$

> **Lem** (Partition of unity reloaded)
> $D(f) \overset{(*)}{=} \bigcup_{i \in I} D(f_i) \Rightarrow \exists I_0 \subseteq I$ finite, $r_i \in R$ $(i \in I_0)$
> $m \in \mathbb{N}$: $f^m = \sum_{i \in I_0} r_i f_i \in R$

$\rightsquigarrow f^m = \sum_p r_p \cdot f_p$ finite sum

On $D(f_p) \cap D(f_q) = D(f_p f_q)$: $\frac{g_p}{f_p} = \varphi = \frac{g_q}{f_q}$

Injectivity above $\Rightarrow \frac{g_p}{f_p} = \frac{g_q}{f_q} \in R_{f_p f_q}$

$$\Rightarrow (f_p f_q)^n (g_p f_q - g_q f_p) = 0 \in R$$

can pick n which works
for all p,q in finite sum

Replace $g_p \rightsquigarrow g_p f_p^n$, $f_p \rightsquigarrow f_p^{n+1}$

Summary $D(f)$ covered by $D(f_p)$ and $\varphi = \frac{g_p}{f_p}$ on $D(f_p)$
and $\underbrace{g_p f_q = g_q f_p}_{(+)}$ $\forall p,q$ in finite sum.

To conclude

$f^r = \sum r_p \cdot f_p$. Set $g = \sum_p r_p \cdot g_p$

$\Rightarrow g \cdot f_q = \sum r_p g_p f_q \overset{(+)}{=} \sum r_p f_p g_q = f^r \cdot g_q \quad \forall q$

$\Rightarrow \frac{g}{f^r} = \frac{g_q}{f_q}$ on $D(f_q)$ $\rightsquigarrow \varphi = \Phi\left(\frac{g}{f^r}\right)$. $\square$

Examples of affine schemes

A) Double points

$R = K[X]/\langle x^2\rangle$, K field $\rightsquigarrow$ $x \in R = \mathcal{O}_{\operatorname{Spec}R}(\operatorname{Spec}R)$ nilpotent element
$\hookrightarrow$ not possible for aff. var.

$\varphi = a + bx \in R \quad (a, b \in K)$

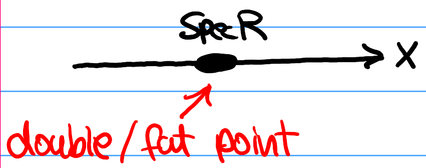
$\varphi^n = a^n + n\cdot a^{n-1}\cdot b x \quad \rightsquigarrow \quad \varphi$ nilpotent $\Leftrightarrow a = 0 \Leftrightarrow \varphi \in \langle x\rangle$

Conversely: $P = \langle x\rangle$ is maximal ideal since $R/\langle x\rangle \cong K$ field

$\Rightarrow \operatorname{Spec}R = \{p\}$ and $K(p) \cong K$

$\varphi(p) = a \in K(p) \quad \rightsquigarrow \quad \varphi$ not uniquely determined by values at pts of $\operatorname{Spec}R$.

Geometric picture

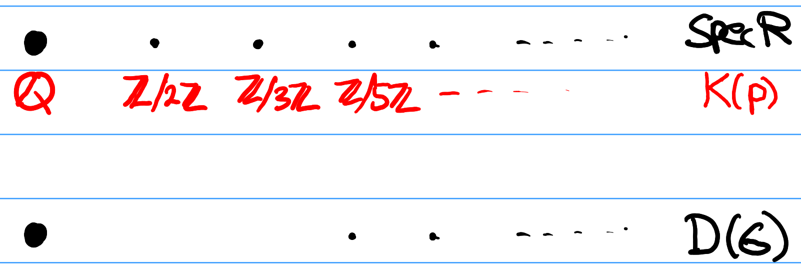


$f \in K[x]$ function

$\Rightarrow f|_{\operatorname{Spec}R}$ only remembers lin. part

(b) Spectrum of the ring of integers

$R = \mathbb{Z} \Rightarrow \operatorname{Spec}R = \{\langle 0\rangle, \langle 2\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}$



$D(6) = \operatorname{Spec}R \setminus V(6) = \operatorname{Spec}R \setminus \{p : \langle 6\rangle \subseteq p\} = \operatorname{Spec}R \setminus \{\langle 2\rangle, \langle 3\rangle\}$

$\hookrightarrow \varphi = 5/6 \in \mathcal{O}_{\operatorname{Spec}R}(D(6))$

$\Rightarrow \varphi(\langle 0\rangle) = \frac{5}{6} \in \mathbb{Q} = K(0)$

$\varphi(\langle p\rangle) = \overline{5}\cdot\overline{6}^{-1} \in \mathbb{Z}/p\mathbb{Z}$ $\rightsquigarrow$ $\varphi(\langle 5\rangle) = 0$, $\varphi(\langle 11\rangle) = \overline{5}\cdot\overline{2} = \overline{10}$

Locally ringed spaces and their morphisms

Have given $X = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ structure of ringed space

$\rightsquigarrow$ would like to say: morphisms $X \xrightarrow{f} Y$ of affine schemes

$= $ morphisms $X \xrightarrow{f} Y$ of ringed spaces

$\Longleftrightarrow$ f pulls back $\varphi \in \mathcal{O}_Y(V)$ to $f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$

Problem definition of $f^*\varphi = \varphi \circ f$ assumed $\varphi : V \longrightarrow K$

Recall:

Chap. 5

Convention From now to Chapter 12 (Schemes):
Assume for every ringed space $(X, \mathcal{F})$ that for $U \subseteq X$ open, we have

$$\mathcal{F}(U) \subseteq \{\varphi : U \to K : \text{function}\}$$

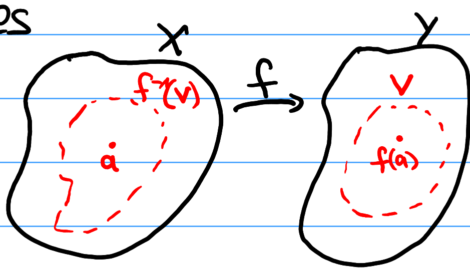
pointwise addition & multiplication

Solution need to include the data $f^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$ as part of morphism $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$.

Rmk $f : X \longrightarrow Y$ morphism of (pre-) varieties

$\rightsquigarrow$ $f^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$
compatible with restrictions

$\rightsquigarrow$ $f_a^* : \mathcal{O}_{Y, f(a)} \longrightarrow \mathcal{O}_{X,a}$ for $a \in X$



$I_{f(a)} = \{\varphi \in \mathcal{O}_{Y,f(a)} : \varphi(f(a)) = 0\} \subseteq \mathcal{O}_{Y,f(a)}$ $\quad$ $I_a = \{\psi \in \mathcal{O}_{X,a} : \psi(a) = 0\} \subseteq \mathcal{O}_{X,a}$

Have:

$$(f_a^*)^{-1}(I_a) = \{\varphi \in \mathcal{O}_{Y,f(a)} : \underbrace{(\varphi \circ f)(a)}_{= \varphi(f(a))} = 0\} = I_{f(a)}$$

inv. image of max. ideal in $\mathcal{O}_{X,a}$ is max. ideal in $\mathcal{O}_{Y,f(a)}$

Def (Locally ringed space)
A <u>locally ringed space</u> is a ringed space $(X, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_{X,p}$ for $p \in X$ is a local ring.

Exa
(a) X pre-variety $\Rightarrow$ X is loc. ringed space
$\ulcorner p \in X$, $U \subseteq X$ affine open nbhd $\Rightarrow \mathcal{O}_{X,p} = A(U)_{I_U(p)} \lrcorner$

(b) $X = \operatorname{Spec} R$ affine scheme $\Rightarrow$ X loc. ringed space
$\ulcorner p \in X \Rightarrow \mathcal{O}_{\operatorname{Spec} R, p} = R_p \lrcorner$

(c) $(X, \mathcal{O}_X)$ loc. ringed space, $U \subseteq X$ open $\Rightarrow (U, \mathcal{O}_X|_U)$ loc. ringed sp.
$\rightsquigarrow$ in particular: open subsets of affine schemes!

Def (Morphisms of locally ringed spaces)
A <u>morphism</u> $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of loc. ringed spaces is a tuple $(f, (f_V^*)_{V \subseteq Y \text{ open}})$ of:

- a continuous map $f: X \longrightarrow Y$
- for every $V \subseteq Y$ open a ring homom. $f_V^*: \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$
  ↑ pull-back on V

Such that
- the pull-back maps are compatible with restrictions:
  $\forall V \subseteq W \subseteq Y$ open, $\varphi \in \mathcal{O}_Y(W)$: $f_V^*(\varphi|_V) = (f_W^* \varphi)|_{f^{-1}(V)} \in \mathcal{O}_X(f^{-1}(V))$
  $\rightsquigarrow f_p^*: \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p} \quad \forall p \in X$

- $\forall p \in X$: $(f_p^*)^{-1} I_p = I_{f(p)}$
  max. ideal in $\mathcal{O}_{X,p}$ — max. ideal in $\mathcal{O}_{Y, f(p)}$

Note Write f_p^*, f_U^* as f^* when clear from context

Exa Morphisms of (pre-)varieties are morph. of loc. ringed spaces.

The correspondence between affine schemes and rings

Have seen

$\text{Affine varieties}_K \xrightarrow{\sim} \text{FinGenRed } K\text{-algebras}$, $X \mapsto A(X)$

Now arbitrary rings!

Pro For any two rings R, S there is a bijection

$$\{\text{morphisms } \operatorname{Spec} R \to \operatorname{Spec} S\} \xleftrightarrow{1:1} \{\text{ring homomorphisms } S \to R\}$$
$$f \xmapsto{\Phi} f^* = f^*_{\operatorname{Spec} S}$$

In particular, we have a natural bijection

$$\{\text{affine schemes}\}/\text{isomorphisms} \xleftrightarrow{1:1} \{\text{rings}\}/\text{isomorphisms}.$$

Pf $\boxed{\Phi \text{ well-defined}}$ $f: \operatorname{Spec} R \to \operatorname{Spec} S \rightsquigarrow f^*_{\operatorname{Spec} S}: \underset{\substack{\cong \\ S}}{\mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} S)} \xrightarrow{\text{ring hom.}} \underset{\substack{\cong \\ R}}{\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)}$

$\boxed{\text{Inverse map } \Psi = \Phi^{-1}}$ $\varphi: S \to R$ ring homomorphism

$\rightsquigarrow$ construct $f = \Psi(\varphi): \operatorname{Spec} R \to \operatorname{Spec} S$
$$p \longmapsto \varphi^{-1}(p)$$

$\boxed{\text{Lem } \varphi: S \to R,\ p \trianglelefteq R \text{ prime} \Rightarrow \varphi^{-1}(p) \trianglelefteq S \text{ prime}}$

→ well-defined map of sets by Lem

→ f continuous:

$J \trianglelefteq S$ ideal, $V(J) = \{q \trianglelefteq S \text{ prime} : J \subseteq q\}$

$$\Rightarrow f^{-1}(V(J)) = \{p \trianglelefteq R \text{ prime} : \underbrace{J \subseteq \varphi^{-1}(p)}_{\Leftrightarrow\ \varphi(J) \subseteq p}\} = V(\varphi(J))$$

↖closed in Spec R

Next Given $V \subseteq \operatorname{Spec} S$ open, need to define

$$f_V^*: \underbrace{\mathcal{O}_{\operatorname{Spec} S}(V)}_{=\{(\sigma_q \in S_q)_{q \in V} : \text{loc. } \frac{g}{f}\}} \longrightarrow \underbrace{\mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V))}_{=\{(\eta_p \in R_p)_{p \in f^{-1}(V)} : \text{loc. } \frac{\tilde{g}}{\tilde{f}}\}}$$

Summary **Given** $\varphi: S \to R$ ring hom. $\rightsquigarrow f: \operatorname{Spec} R \to \operatorname{Spec} S$
$p \mapsto \varphi^{-1}(p)$

Want

$$f_V^*: \underbrace{\mathcal{O}_{\operatorname{Spec} S}(V)}_{=\{(\sigma_q \in S_q)_{q \in V}:\ \text{loc. } \frac{g}{f}\}} \longrightarrow \underbrace{\mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V))}_{=\{(\eta_p \in R_p)_{p \in f^{-1}(V)}:\ \text{loc. } \frac{\tilde{g}}{\tilde{f}}\}} \quad \text{for } V \subseteq_{\text{open}} \operatorname{Spec} S$$

$\rightsquigarrow$ Given $(\sigma_q)_{q \in V}$ need to specify $f_V^*((\sigma_q)_{q \in V})_p$ for $p \in f^{-1}(V)$!

Lem $\varphi: S \to R$ ring hom., $p \trianglelefteq R$ prime

$\Rightarrow$ $\varphi_p: S_{\varphi^{-1}(p)} \longrightarrow R_p$, $\frac{g}{f} \longmapsto \frac{\varphi(g)}{\varphi(f)}$ well-def. ring hom.
$\varphi_p^{-1}\ p.R_p = \varphi^{-1}(p).S_{\varphi^{-1}(p)}$

Def
$$f_V^*: \mathcal{O}_{\operatorname{Spec} S}(V) \longrightarrow \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V))$$
$$(\sigma_q)_{q \in V} \longmapsto (\eta_p = \varphi_p(\sigma_{\varphi^{-1}(p)}))_{p \in f^{-1}(V)}$$

Check
- $\sigma_q = \frac{g}{f}$ locally $\Rightarrow$ $\eta_p = \frac{\varphi(g)}{\varphi(f)}$ locally $\rightsquigarrow$ well-def.
- f_V^* ring hom., compatible with restrictions
- $f_p^* = \varphi_p$ map on stalks $\rightsquigarrow$ map of local rings

$\Rightarrow$ $(f, (f_V^*)_{V \subseteq \operatorname{Spec} S \text{ open}}): \operatorname{Spec} R \to \operatorname{Spec} S$ map of loc. ringed spaces

Exercise check Φ, Ψ are inverse maps $\square$

Fancy category version

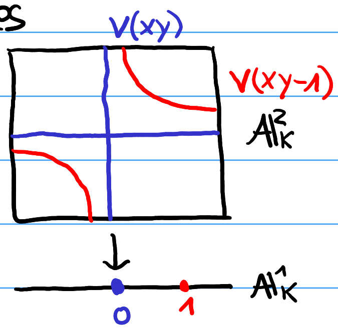
Contravariant functor Affine Schemes $\longrightarrow$ Rings
$$X \longmapsto R = \mathcal{O}_X(X)$$
$$(X \xrightarrow{f} Y) \longmapsto (\mathcal{O}_Y(Y) \xrightarrow{f^*} \mathcal{O}_X(X))$$

is an equivalence of categories.

Examples of morphisms of affine schemes

See also [Vakil, Sect. 3.2.10, 3.3]

Exa 1 For $K=\overline{K}$ consider morphism of affine varieties



$$\mathbb{A}^2_K \xrightarrow{f} \mathbb{A}^1_K, \quad (x,y) \mapsto x\cdot y$$

What is the analogous morphism of affine schemes?

$$K[x,y] = A(\mathbb{A}^2_K) \xleftarrow{f^*} A(\mathbb{A}^1_K) = K[t]$$
$$xy \longleftarrow\!\shortmid\; t$$

$\Rightarrow$ map $\varphi = f^*: K[t] \to K[x,y]$ of rings

$\updownarrow$

morph. $F: \operatorname{Spec} K[x,y] \to \operatorname{Spec} K[t]$ of affine schemes

Map of sets?

- $(a,b) \in \mathbb{A}^2_K \rightsquigarrow m_{(a,b)} = \langle x-a, y-b \rangle \in \operatorname{Spec} K[x,y]$

$$F(m_{(a,b)}) = \varphi^* m_{(a,b)} = \{\eta(t) \in K[t] : \underbrace{\varphi(\eta)}_{\eta(xy)} \in m_{(a,b)}\}$$

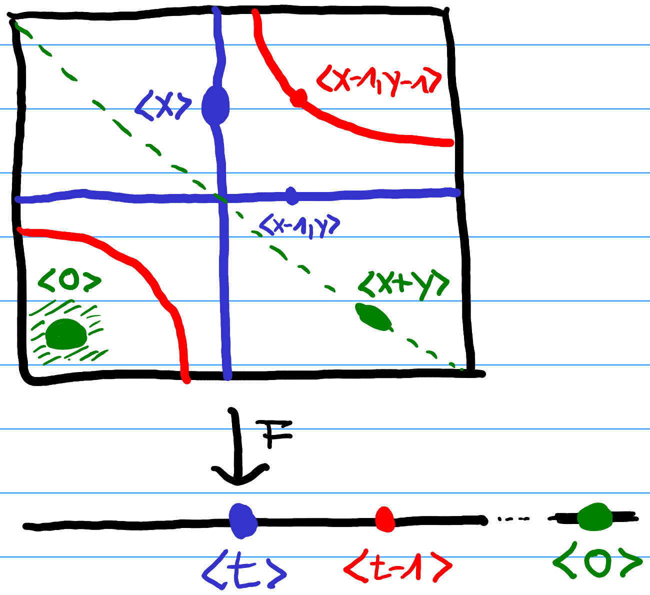
$$= \{\eta(t) \in K[t] : \eta(ab) = 0\}$$

$$= \langle t - ab \rangle = m_{ab}$$

$\Rightarrow f(a,b) = ab \overset{\text{translates}}{\rightsquigarrow} F(m_{(a,b)}) = m_{ab}$

- $F(\langle x \rangle) = \varphi^* \langle x \rangle = \{\eta(t) \in K[t] : \eta(xy) \in \langle x \rangle\}$

$$= \{\eta(t) \in K[t] : \eta(0) = 0\} = m_0 = \langle t \rangle$$



Spec K[x,y]

Spec K[t]

- $F(\langle 0\rangle) = \varphi^*\langle 0\rangle = \{\eta(t)\in K[t] : \eta(xy)=0\} = \langle 0\rangle$
- $F(\langle x+y\rangle) = \varphi^*\langle x+y\rangle = \{\eta(t)\in K[t] : \eta(xy)\in \underbrace{\langle x+y\rangle}_{=I(\{(x,-x):\, x\in K\})}\}$

$$= \{\eta(t)\in K[t] : \underbrace{\eta(x\cdot(-x))=0 \ \forall x\in K}_{\eta = a_0+a_1t+a_2t^2+\dots \Rightarrow \eta(-x^2)=a_0-a_1x^2+a_2x^4+\dots}\} = \langle 0\rangle$$

More generally

$X \xrightarrow{f} Y$ map of aff. var. $/K$ $\rightsquigarrow \varphi = f^* : A(Y)\to A(X)$

$\rightsquigarrow F : \operatorname{Spec} A(X) \to \operatorname{Spec} A(Y)$

For $p\in \operatorname{Spec} A(X) \rightsquigarrow Z_p = V_X(p)\subseteq X$ irreducible subvariety

$\rightsquigarrow f(Z_p)\subseteq Y$ irreducible [Exercise 2.22(b)]

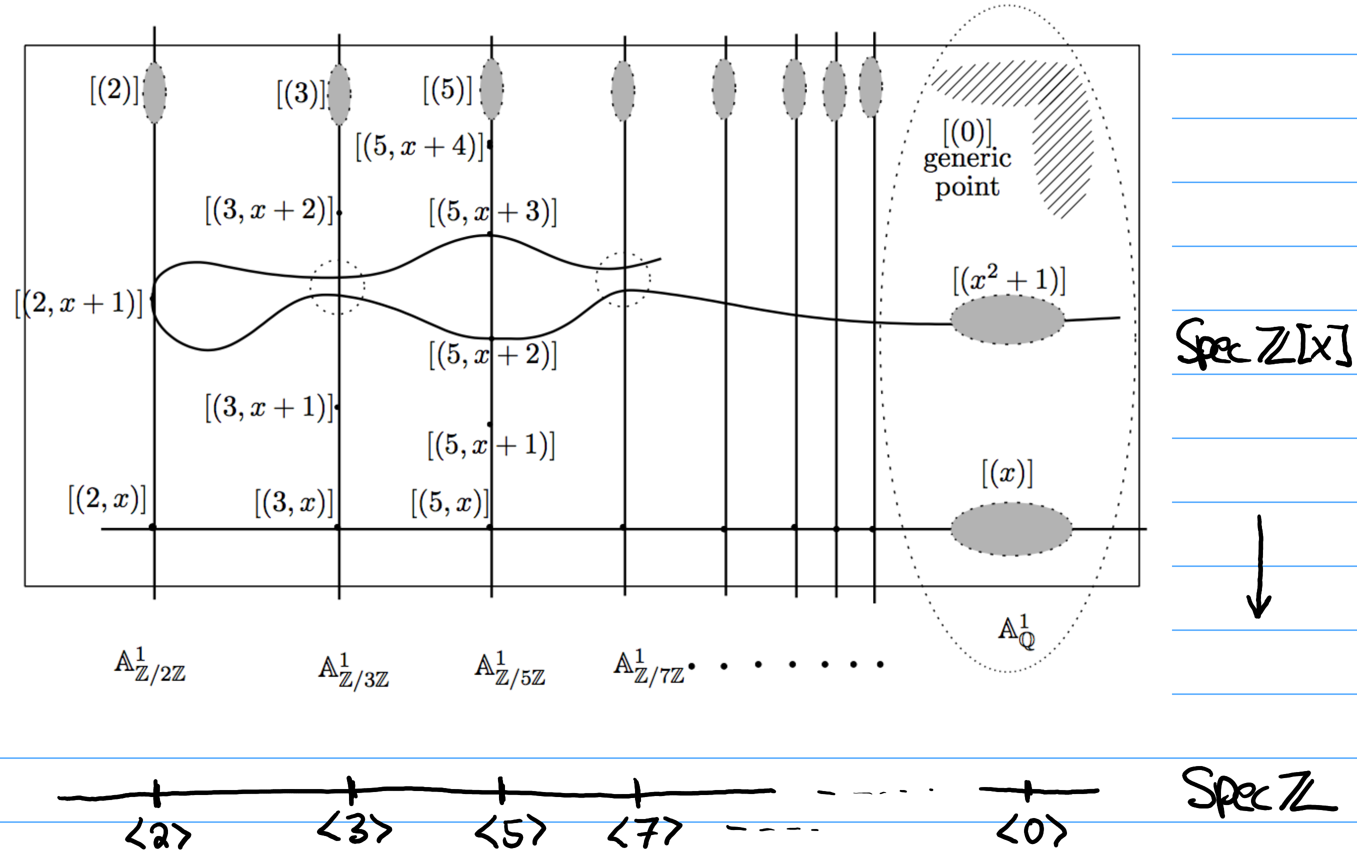
$\rightsquigarrow Z' = \overline{f(Z_p)} \subseteq Y$ irreducible subvar. [Ex. 2.20]

$\rightsquigarrow q = I_Y(Z') \trianglelefteq A(Y)$ prime ideal

$\Rightarrow F : \operatorname{Spec} A(X) \to \operatorname{Spec} A(Y)$, $p \mapsto q$.

Exa 2 (The affine line over $\mathbb{Z}$)

Ring map $\mathbb{Z} \to \mathbb{Z}[x] \rightsquigarrow \operatorname{Spec} \mathbb{Z}[x] \longrightarrow \operatorname{Spec} \mathbb{Z}$



from: [Mumford-Oda, Algebraic Geometry II]

found via: pbelmans.ncag.info/blog/atlas

Exercise Meditate over this picture.

Affine subschemes

Have seen

X affine variety $\quad Y \subseteq X$ closed $\rightsquigarrow$ Y affine variety

$\rightsquigarrow$ What about X affine scheme?

Idea $\quad Y \subseteq X$ closed $\hat{=}$ $J = I(Y) \subseteq A(X)$ radical ideal
and $A(Y) = A(X)/J$ with map $Y \to X \;\hat{=}\; A(Y) \leftarrow A(X)$

↑quotient map

$\rightsquigarrow$ X affine scheme: get rid of radical!

Construction (Affine subschemes)
R ring, $X = \operatorname{Spec} R$. Then an affine subscheme of X is a morphism

$$Y = \operatorname{Spec} R/J \xrightarrow{f} \operatorname{Spec} R = X \quad \text{assoc. to } R \xrightarrow{\varphi} R/J$$

for $J \trianglelefteq R$ any ideal.

Rmks (a) $\boxed{\text{Lem } \{\text{prime ideals } q \trianglelefteq R/J\} \xrightarrow[\sim]{q \mapsto \varphi^{-1}(q)} \{\text{prime ideals } p \trianglelefteq R \text{ with } J \subseteq p\} = V(J) \subseteq \operatorname{Spec} R}$

$\Rightarrow f$ injective, $f(Y) = V(J)$ $\leftarrow$ $\operatorname{Spec} R/J$ gives struct. of affine scheme to $V(J)$.

(b) $\varphi: R \to R/J$ surjective.
Conversely $\psi: R \to S$ surjective ring hom. $\overset{\text{Isom. Thm.}}{\Longrightarrow}$ $S \cong R/\ker(\psi)$

Thus: affine subschemes $Y \xrightarrow{f} X$ = morph of aff. schemes with $f^*: \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ surjective.

(c) $\{\text{ideals in } R\} \overset{1:1}{\longleftrightarrow} \{\text{affine subschemes of } \operatorname{Spec} R\}$

$$J \longmapsto (\operatorname{Spec}(R/J) \to \operatorname{Spec}(R))$$

Scheme-theoretic intersections and unions

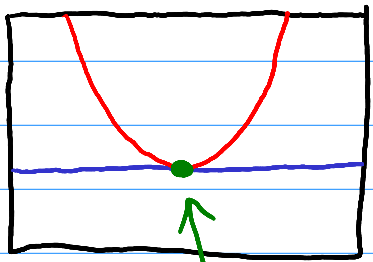
Let $\operatorname{Spec} R/J_1 \to \operatorname{Spec} R$, $\operatorname{Spec} R/J_2 \to \operatorname{Spec} R$ be affine subschemes.

$$\operatorname{Spec} R/J_1 \cap \operatorname{Spec} R/J_2 := \operatorname{Spec} R/(J_1+J_2)$$ no radical

$$\operatorname{Spec} R/J_1 \cup \operatorname{Spec} R/J_2 := \operatorname{Spec} R/(J_1 \cap J_2)$$

Note Gives correct underlying sets: $V(J_1)\cap V(J_2) = V(J_1+J_2)$
$V(J_1)\cup V(J_2) = V(J_1\cap J_2)$

Exa



$\operatorname{Spec} K[x,y]$

$\operatorname{Spec} K[x,y]/\langle y-x^2\rangle = X_1$

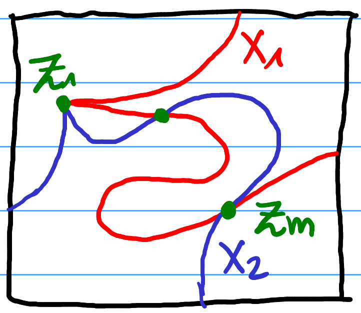
$\operatorname{Spec} K[x,y]/\langle y\rangle = X_2$

$X_1\cap X_2 = \operatorname{Spec} K[x,y]/\langle y-x^2, y\rangle \cong \operatorname{Spec} K[x]/\langle x^2\rangle$

$$\Rightarrow \dim_K \mathcal{O}_{X_1\cap X_2}(X_1\cap X_2) = \dim_K K[x]/\langle x^2\rangle = 2$$

intersection multiplicity of X_1, X_2 at $(0,0)$.

More generally



$X_1, X_2 \subseteq \mathbb{A}^2_K$ curves not sharing any irred. components

$\rightsquigarrow \mathcal{X}_i = \operatorname{Spec} K[x,y]/I(X_i) \to \operatorname{Spec} K[x,y]$ affine subschemes

$\rightsquigarrow \mathcal{X}_1 \cap \mathcal{X}_2 = Z_1 \sqcup \dots \sqcup Z_m$ connected components

$\rightsquigarrow \dim_K \mathcal{O}_{Z_i}(Z_i) =$ intersect. multiplicity of X_1, X_2 at Z_i

Distinguished opens as affine schemes

Have seen X affine variety, $f \in A(X) \rightsquigarrow D(f)$ affine var.

$A(D(f)) = A(X)_f$

$\rightsquigarrow$ What about affine schemes?

Pro (Distinguished open sets are affine schemes)

R ring, $f \in R \Rightarrow$ For the ring homomorphism $R \xrightarrow{\varphi} R_f$, the associated morphism

$$\operatorname{Spec} R_f \xrightarrow{i_f} \operatorname{Spec} R$$

is an open embedding with image $D(f)$: $\operatorname{Spec} R_f \cong (D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)})$.

Pf $\operatorname{Spec} R_f \overset{\text{Lem}}{=\!=} \{p \in \operatorname{Spec} R : f \notin p\} = D(f)$ as sets
$pR_f \leftarrow\!\shortmid p$

$\Rightarrow i_f$ injective, continuous with image $D(f)$

To see $i_f^{-1} : D(f) \longrightarrow \operatorname{Spec} R_f$ continuous:

$V(J) \subseteq \operatorname{Spec} R_f$ closed, $J \trianglelefteq R_f$ ideal $\Rightarrow \tilde{J} = \varphi^{-1}(J) \trianglelefteq R$ ideal

and $i_f(V(J)) \overset{\text{Ex.}}{=} V(\tilde{J}) \cap D(f)$

$\Rightarrow i_f$ closed, so homeomorphism onto $D(f)$.

Comparison of structure sheaves

$U \subseteq D(f)$ open $\rightsquigarrow$ Want: $i_f^* : \underbrace{\mathcal{O}_{D(f)}(U)}_{= \mathcal{O}_{\operatorname{Spec} R}(U)} \longrightarrow \mathcal{O}_{\operatorname{Spec} R_f}(U)$ isom.

Claim suffices to prove this for basis $U = D(fg)$ of opens in $D(f)$

Pf Sheaf gluing axioms: sections on U = sect. on cover $\{D(fg_i) : i \in I\}$ of U that agree on further covers of $D(fg_i) \cap D(fg_j)$

$\Rightarrow i^*_{f,D(fg_i)}$ isomorph. implies $i^*_{f,U}$ isom. $\forall U \subseteq D(f)$ open. ※

$U = D(f \cdot g) \rightsquigarrow \mathcal{O}_{D(f)}(D(fg)) = R_{fg}$
$\rightsquigarrow \mathcal{O}_{\operatorname{Spec} R_f}(D(fg)) = (R_f)_{fg} \cong R_{fg}$
} map i_f^* = identity.

$\square$

Definition of schemes

<u>Have seen</u> Prevariety = ringed space which has finite open cover by affine varieties.

Def (Schemes)
A <u>scheme</u> is a locally ringed space that has an open cover by affine schemes.
Morphisms of schemes = morphisms of loc. ringed spaces.

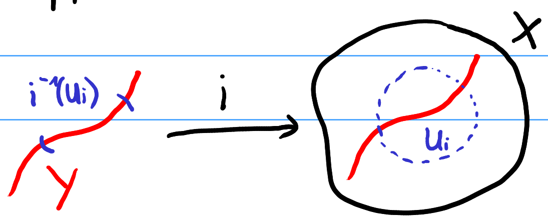
<u>Rmks</u> (a) no finiteness assumption
$\rightsquigarrow$ (affine) schemes are not necess. Noetherian
(b) had (prevarieties $\rightsquigarrow$ varieties), could have: (preschemes $\rightsquigarrow$ schemes)
instead: (schemes $\rightsquigarrow$ separated schemes).

<u>Construction</u> (Open & closed subschemes)
X scheme
(a) $U \subseteq X$ open $\rightsquigarrow$ $\exists$ cover $\{U_i : i \in I\}$ of X by aff. schemes
$\rightsquigarrow$ each $U_i \cap U$ has cover $\{D_{U_i}(f_{ij}) : j \in J_i\}$ by disting. opens in U_i
$\Rightarrow$ $\{D_{U_i}(f_{ij}) : i \in I, j \in J_i\}$ cover of U by aff. schemes
$\Rightarrow$ $(U, \mathcal{O}_U)$ is scheme itself
$\nwarrow$ <u>open subscheme</u> of X.

(b) <u>Closed subscheme</u> of X = morphism $i : Y \longrightarrow X$ (Y scheme)
such that $\exists^*$ open cover $\{U_i : i \in I\}$ of X by aff. schemes:

$$i\big|_{i^{-1}(U_i)} : i^{-1}(U_i) \longrightarrow U_i \quad \text{is affine subscheme } \forall i \in I$$

$*$: in this case: holds for <u>any</u> open cover by aff. schemes U_i



Schemes from prevarieties

Big picture

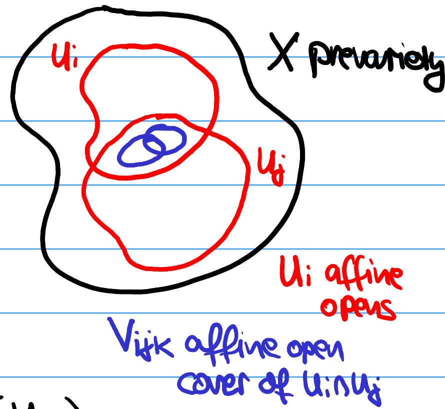
- have developed theory of (pre-) varieties, many examples
- new definition of schemes $\rightsquigarrow$ have to do everything again?
- No: X prevariety $\rightsquigarrow$ X_{sch} scheme

Idea 1 (Gluing)

X prevariety

$\rightsquigarrow$ $\{U_i : i \in I\}$ affine open cover of X

$\rightsquigarrow$ $\{V_{ijk} : k \in J_{ij}\}$ affine open cover of $U_i \cap U_j$



$\Rightarrow$ get X_{sch} by gluing together

$$U_{i,sch} = \operatorname{Spec} A(U_i) \text{ along } V_{ijk,sch} = \operatorname{Spec} A(V_{ijk}).$$

Know maps $A(U_i) \to A(V_{ijk})$

Idea 2 (Just write down answer)

Pro (Prevarieties as schemes)

(a) X prevariety over $K = \overline{K}$

$\Rightarrow X_{sch} = \{[Y] : Y \subseteq X \text{ irreducible closed subset}\}$ — set

$Z \subseteq X$ closed $\Rightarrow \mathcal{Z} = \{[Y] \in X_{sch} : Y \subseteq Z\}$ closed sets of X_{sch} — topology

$U \subseteq X$ open $\rightsquigarrow \mathcal{U} = X_{sch} \setminus \mathcal{Z}$ open sets of X_{sch} — structure sheaf

$Z = X \setminus U$ $\quad \mathcal{O}_{X_{sch}}(\mathcal{U}) := \mathcal{O}_X(U)$ with same restrict.

Then $(X_{sch}, \mathcal{O}_{X_{sch}})$ is a scheme

(b) $X \xrightarrow{f} Y$ morphism of prevarieties $\rightsquigarrow X_{sch} \xrightarrow{f_{sch}} Y_{sch}$ morphism of schemes.

<u>Pf idea</u> (a) Follow gluing recipe from Idea 1 above

$U_i \subseteq X$ open affine variety $\rightsquigarrow$ $\mathfrak{p} \in \operatorname{Spec} A(U_i) \;\hat{=}\; Y \subseteq U_i$ irred. closed subset

$\quad\updownarrow$

$\overline{Y} \subseteq X$ irred. closed subset

$\Rightarrow [\overline{Y}] \in X_{sch}$.

<u>Check</u> $\bigcup_{i \in I} \operatorname{Spec} A(U_i) \longrightarrow X_{sch}$ gives open cover

$\rightsquigarrow$ gluing recipe gives X_{sch} as top. space

To finish proof:

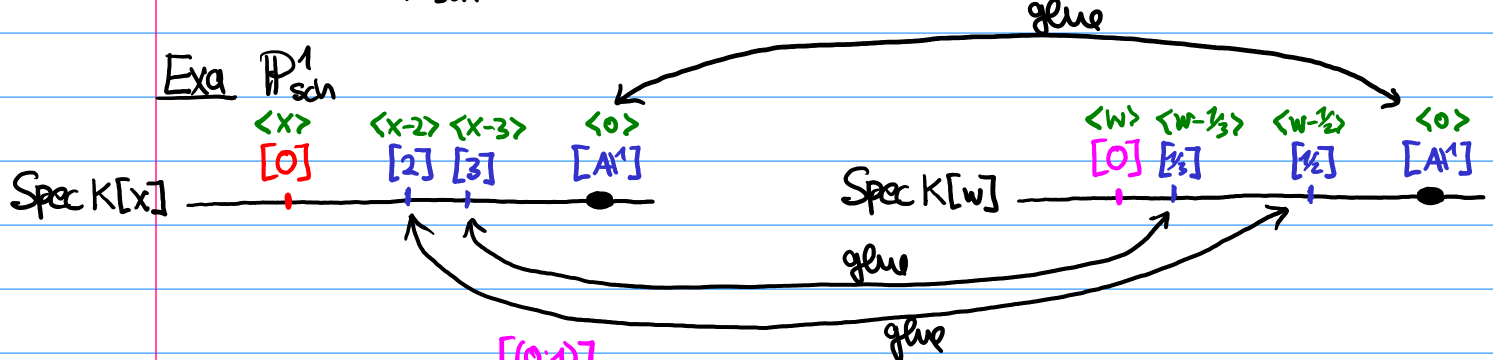
- $\mathcal{O}_X(U) = \mathcal{O}_{X_{sch}}(U)$ $\rightsquigarrow$ first check on affine varieties
- For morphism $X \xrightarrow{f} Y$: cover Y by aff. subsets V, cover $f^{-1}(V)$ by aff. subsets U $\Big\}$ $f^*: \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$

  Show that maps

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_X(U) & \longrightarrow & \operatorname{Spec} \mathcal{O}_Y(V) \\ \cap\!\mid & & \cap\!\mid \\ X_{sch} & \xrightarrow{\exists f_{sch}} & Y_{sch} \end{array}$$

glue together. $\square$

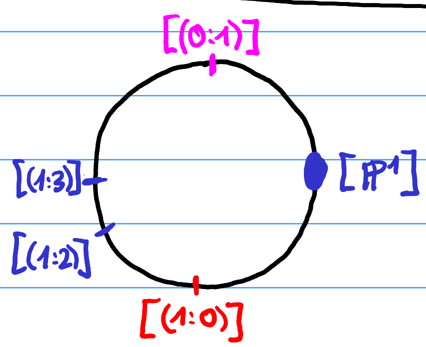
<u>Exa</u> $\mathbb{P}^1_{sch}$



gluing: $\operatorname{Spec} K[x]_x \longrightarrow \operatorname{Spec} K[w]_w$

$1/x \longleftarrow\!\!\mid w$

$\rightsquigarrow \mathbb{P}^1_{sch}$



<u>Note</u> (for category fans)

$\text{Prevarieties}_K \longrightarrow \text{Schemes}_K$ is a fully faithful functor

bijection: $\operatorname{Mor}(X,Y) \xrightarrow{\sim} \operatorname{Mor}(X_{sch}, Y_{sch})$

$X \longmapsto X_{sch}$

$(X \xrightarrow{f} Y) \longmapsto (X_{sch} \xrightarrow{f_{sch}} Y_{sch})$

$\Rightarrow \text{Prevarieties}_K \subseteq \text{Schemes}_K$ full subcategory

explain this more below!

Properties of schemes: finite type and reduced schemes

Big question in rest of chapter

$$\boxed{\text{Which schemes } Y \text{ are of the form } Y = X_{sch} \text{ for } X \text{ a (pre-) variety } / K = \bar{K}\,?}$$

Some ideas

- prevarieties glued together from finitely many affine varieties
- X affine variety $\Rightarrow$ $A(X)$ is finitely generated K-algebra & reduced
- Gluing maps $X \to Y$ corresp. to $A(Y) \to A(X)$ K-algebra homomorph.
- X variety $\rightsquigarrow$ X separated, i.e. $\Delta_X \subseteq X \times X$ closed

Goal generalize these properties for schemes

Def (Properties of schemes)

(a) Y scheme. A scheme over Y is a pair (X, f) of a scheme X and a morphism $f: X \to Y$.

A morphism of schemes $X_1 \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{f_2} Y$ over Y is a morphism $g: X_1 \to X_2$ such that $f_1 = f_2 \circ g$.

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

$X \to \operatorname{Spec} S$: scheme over S

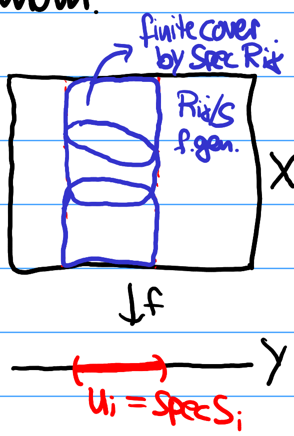
If $X = \operatorname{Spec} R$: equiv. to $S \to R$ ring homom.

(b) $X \xrightarrow{f} Y$ is of finite type over Y if

$\exists$ open cover $\{U_i = \operatorname{Spec} S_i : i \in I\}$ of Y

$\exists$ finite open cover $\{\operatorname{Spec} R_{ij} : j \in J_i\}$ of $f^{-1}(U_i)$

such that induced homom. $S_i \to R_{ij}$ makes R_{ij} a finitely generated S_i-algebra



(c) X is reduced if the rings $\mathcal{O}_X(U)$ have no nonzero nilpotent elements $\forall U \subseteq X$ open.

Exercise X scheme. Show the following are equivalent:
(i) X is reduced, i.e. $\forall U \subseteq X$ open, $\mathcal{O}_X(U)$ is a reduced ring
(ii) $\exists$ open cover $\{U_i = \operatorname{Spec} R_i : i \in I\}$ of X with R_i reduced
(iii) $\forall p \in X$ the local ring $\mathcal{O}_{X,p}$ is reduced.

Rmk K field, $X \xrightarrow{f} \operatorname{Spec} K = \{\langle 0 \rangle\}$ scheme over K
$\rightsquigarrow$ For $U \subseteq X$ open: $f_U^*: K \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ a K-algebra
$\Rightarrow U = \operatorname{Spec} R \subseteq X$ affine open $\Rightarrow R$ is K-algebra.

X finite type over K $\rightsquigarrow$ X has fin. cover by $\operatorname{Spec} R_i$, R_i fin. gen. K-algebra
X reduced $\rightsquigarrow$ R_i is reduced

Cor Y scheme, $K = \overline{K}$ field
$Y = X_{sch}$ for X a prevar. over $K \iff Y$ is reduced and of finite type over K
Moreover: $\operatorname{Mor}_{\text{schemes}_K}(X_{sch}, Y_{sch}) \cong \operatorname{Mor}_{\text{prevar.}}(X, Y)$
$$f_{sch} \leftarrow\!\shortmid f$$

Pf idea
$\{U_i = \operatorname{Spec} R_i : i \in I\}$ fin. cover of Y as above $\Rightarrow R_i = A(X_i)$, X_i aff. var.
$\rightsquigarrow X_i$ glue to prevariety X $\qquad U_i = (X_i)_{sch}$
and $X_{sch} = Y$.

$(f \mapsto f_{sch}$ injective$)$:
$$\begin{array}{ccc} X & \xhookrightarrow{a \mapsto [a]} & X_{sch} \\ f\downarrow & & \downarrow f_{sch} \\ Y & \hookrightarrow & Y_{sch} \end{array}$$
diagram of sets $\Rightarrow f_{sch}$ determines f.

$(f \mapsto f_{sch}$ surjective$)$:
Given $g: X_{sch} \to Y_{sch}$ choose covers by affine schemes
$\rightsquigarrow$ induced maps of affine var. glue to $X \xrightarrow{f} Y$ w/ $g = f_{sch}$.
$\square$

The fiber product

<u>Would like</u>: X scheme is separated if $\Delta_X \longrightarrow X \times X$ closed subscheme

problem: what are these?

How to define $X \times Y$ for X, Y schemes?

<u>Naive attempt</u>: take set product $X \times Y$, put topology, $\mathcal{O}_{X \times Y}, \ldots$

$\rightsquigarrow$ Problem: $\mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k$ as varieties

$$\underbrace{(\mathbb{A}^1_k)_{sch}}_{=\{[a] : a \in \mathbb{A}^1_k\} \cup \{[\mathbb{A}^1_k]\}} \times (\mathbb{A}^1_k)_{sch} = \{([a],[b]) : (a,b) \in \mathbb{A}^2_k\} \cup \{([a],[\mathbb{A}^1_k]) : a \in \mathbb{A}^1_k\}$$
$$\cup \{([\mathbb{A}^1_k],[b]) : b \in \mathbb{A}^1_k\} \cup \{([\mathbb{A}^1_k],[\mathbb{A}^1_k])\}$$

set product.

$\Rightarrow$ lots of stuff from $(\mathbb{A}^2_k)_{sch}$ missing

e.g. $[V(x-y)]$.

<u>Less naive attempt</u> X, Y affine varieties over $k = \bar{k}$ $\Rightarrow A(X \times Y) = A(X) \otimes_k A(Y)$.

Maybe: $(\operatorname{Spec} R) \times (\operatorname{Spec} S) = \operatorname{Spec}(R \otimes S)$?

tensor product over which ring?

<u>Def</u> (Affine fiber product)

$X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ affine schemes over $Z = \operatorname{Spec} A$ $\rightsquigarrow$ R, S A-algebras

$$\Longrightarrow \operatorname{Spec} R \times_{\operatorname{Spec} A} \operatorname{Spec} S := \operatorname{Spec} R \otimes_A S.$$

Have commutative diagram:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \quad \widehat{=} \quad \begin{tikzcd} R \otimes_A S & S \arrow[l, "1 \otimes s \mapsfrom s"'] \\ R \arrow[u, "r \otimes 1 \mapsfrom r"] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$\underline{\text{Exa}}$ (a) X, Y aff. var. $/K$ $\rightsquigarrow$ $X_{sch} = \operatorname{Spec} A(X)$, $Y_{sch} = \operatorname{Spec} A(Y)$ schemes $/K$

$\Rightarrow X_{sch} \times_{\operatorname{Spec} K} Y_{sch} = \operatorname{Spec} A(X) \otimes_K A(Y) = \operatorname{Spec} A(X \times Y) = (X \times Y)_{sch}$.

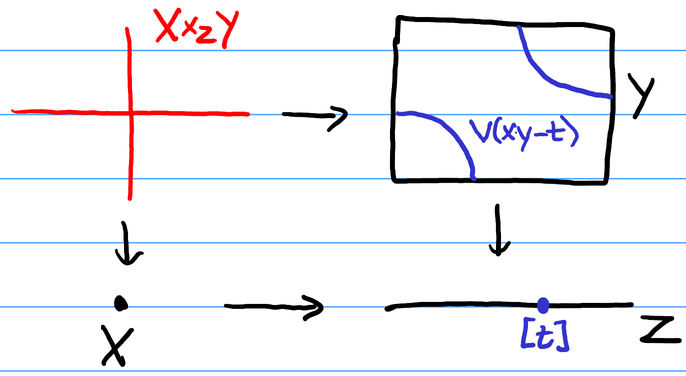
(b)

$X = \operatorname{Spec} K \to Z = \operatorname{Spec} K[t]$

$0 \leftarrow\!\shortmid t$

$Y = \operatorname{Spec} K[x,y] \to Z = \operatorname{Spec} K[t]$

$x \cdot y \leftarrow\!\shortmid t$



$\Rightarrow X \times_Z Y = \operatorname{Spec}(K \otimes_{K[t]} K[x,y])$ $\rightsquigarrow$ see $K = K[t]/\langle t \rangle$

$\cong \operatorname{Spec}(K[x,y]/\langle xy \rangle)$

$$\boxed{\begin{array}{l} \underline{\text{Lem}}\ R \xrightarrow{\varphi} S \text{ ring hom., } J \trianglelefteq R \text{ ideal} \\ \Rightarrow (R/J) \otimes_R S \cong S/\underbrace{\langle \varphi(J) \rangle}_{\text{ideal in } S} \end{array}}$$

Going beyond affine schemes?

$\underline{\text{Construction}}$ (Fiber Products)

$X \xrightarrow{f_X} Z$ and $Y \xrightarrow{f_Y} Z$ schemes $/Z$.

A $\underline{\text{fiber product}}$ of X, Y over Z is a tuple (P, π_X, π_Y) of

- a scheme P
- morphisms $\pi_X : P \to X$, $\pi_Y : P \to Y$

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists! g"] \arrow[drr, bend left, "h_Y"] \arrow[ddr, bend right, "h_X"'] & & \\ & P \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "f_Y"] \\ & X \arrow[r, "f_X"] & Z \end{tikzcd}$$

such that: $\forall$ scheme W, morph. $W \xrightarrow{h_X} X$, $W \xrightarrow{h_Y} Y$ with $f_X \circ h_X = f_Y \circ h_Y$ $\exists$ unique morph. $W \xrightarrow{g} P$ such that $h_X = \pi_X \circ g$, $h_Y = \pi_Y \circ g$.

Pro (Existence & uniqueness of fiber products)

(a) X, Y, Z affine schemes $\Rightarrow$ $X \times_Z Y$ above is a fiber product

(b) X, Y, Z arbitrary schemes $\Rightarrow$ Fiber product P exists & is unique / isomorphism.

Pf sketch

(a) $\left.\begin{array}{l} W = \mathrm{Spec}(B) \\ X = \mathrm{Spec}(R) \\ Y = \mathrm{Spec}(S) \\ Z = \mathrm{Spec}(A) \end{array}\right\}$

$$\begin{tikzcd}
B & R\otimes_A S \arrow[l, dashed, "\varphi\otimes\psi"'] & S \arrow[l] \arrow[ll, bend right, "\psi"'] \\
& R \arrow[ul, "\varphi"] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

$\rightsquigarrow$ universal prop. of tensor product $R \otimes_A S$.

W arbitrary $\rightsquigarrow$ cover by schemes $\mathrm{Spec}(B_i)$, morphisms $\mathrm{Spec}(B_i) \longrightarrow \mathrm{Spec}(R \otimes_A S)$ glue to $W \longrightarrow \mathrm{Spec}(R \otimes_A S)$

(b) X, Y, Z arbitrary $\rightsquigarrow$ choose compatible affine covers

$\rightsquigarrow$ glue affine fiber products

(see [Pro 5.15] $\leftrightsquigarrow$ video 05.05) #

Notation $P = X \times_Z Y$ also for X, Y, Z schemes.

Def (Scheme-theoretic intersection)

$X_1 \xrightarrow{i_1} Y$ and $X_2 \xrightarrow{i_2} Y$ closed subschemes

$\Longrightarrow$ $X_1 \cap X_2 := X_1 \times_Y X_2 \xrightarrow[= i_2 \circ \pi_{X_2}]{i_1 \circ \pi_{X_1}} Y$ scheme-theoretic intersection

$\rightsquigarrow$ closed subscheme $i : X_1 \cap X_2 \to Y$

$U = \mathrm{Spec}\, R \subseteq Y$ affine open

$\Rightarrow$ $i_1^{-1}(U) = \mathrm{Spec}\, R/J_1 \to U$, $i_2^{-1}(U) = \mathrm{Spec}\, R/J_2 \to U$, $J_1, J_2 \trianglelefteq R$

and

$$i^{-1}(U) = \mathrm{Spec}\, R/J_1 \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R/J_2 = \mathrm{Spec}(R/J_1 \otimes_R R/J_2)$$
$$= \mathrm{Spec}(R/(J_1 + J_2))$$

Separatedness and varieties as schemes

Have seen

$\text{Prevarieties}_K \subseteq \text{Schemes}$ as reduced, finite type schemes $/K$

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta_X"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\ & X\times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

Def (Separated schemes) X scheme over Y

(a) The <u>diagonal morphism</u> $\Delta_X : X \to X\times_Y X$ is the morphism induced by (id_X, id_X) from the univ. property of $X\times_Y X$.

(b) X is <u>separated</u> over Y if Δ_X is a closed embedding.

$\Longleftrightarrow$ image of Δ_X closed in $X\times_Y X$.

Pro (Varieties as schemes)

K alg. closed field. Then there is a bijection

$$\{\text{varieties over } K\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{separated, reduced schemes} \\ \text{of finite type over } K\end{array}\right\}$$

$$X \longmapsto X_{sch}$$

inducing a bijection $Mor(X,Y) \xrightarrow{\sim} Mor_{sch_K}(X_{sch}, Y_{sch})$.

Convention Identify variety X with its scheme X_{sch}.

→ "variety" = separated, reduced schemes of fin. type $/K=\overline{K}$

→ "points of a variety" = closed points $[y_0] \in X_{sch}$.

→ "morphisms of varieties" = morphisms over K.

Exa (a) $\mathbb{A}^n_K = \operatorname{Spec} K[x_1, \dots, x_n]$ for $K = \overline{K}$

(b) Complex conjugation $\varphi : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$, $f \mapsto \overline{f}$ is ring homom., but <u>not</u> $\mathbb{C}$-algebra morphism.

$\rightsquigarrow$ $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$ induced by φ is scheme morph. but <u>not</u> morph. of varieties.